Chapter 7. Covariant Formulation of Electrodynamics

Notes:

- *Most of the material presented in this chapter is taken from Jackson, Chap. 11, and Rybicki and Lightman, Chap. 4.*
- *Starting with this chapter, we will be using Gaussian units for the Maxwell equations and other related mathematical expressions.*
- *In this chapter, Latin indices are used for space coordinates only (e.g., $i = 1,2,3$, etc.), while Greek indices are for space-time coordinates (e.g., $\alpha = 0,1,2,3$, etc.).*

7.1 The Galilean Transformation

Within the framework of Newtonian mechanics, it seems natural to expect that the velocity of an object as seen by observers at rest in different inertial frames will differ depending on the relative velocity of their respective frame. For example, if a particle of mass m has a velocity $\mathbf{u}'$ relative to an observer who is at rest an inertial frame K', while this frame is moving with at a constant velocity $\mathbf{v}$ as seen by another observer at rest in another inertial frame K, then we would expect that the velocity $\mathbf{u}$ of the particle as measured in K to be

$$\mathbf{u} = \mathbf{u}' + \mathbf{v}. \tag{7.1}$$

That is, it would seem reasonable to expect that velocities should be added when transforming from one inertial frame to another; such a transformation is called a **Galilean transformation**. In fact, it is not an exaggeration to say that this fact is at the heart of Newton's Second Law. Indeed, if we write the mathematical form of the Second Law in frame K we have

$$\begin{aligned} \mathbf{F} &= m\frac{d^2\mathbf{x}}{dt^2} = m\frac{d\mathbf{u}}{dt} \\ &= m\frac{d\left(\mathbf{u}' + \mathbf{v}\right)}{dt}, \end{aligned} \tag{7.2}$$

but since $\mathbf{v}$ is constant

$$\mathbf{F} = m\frac{d\mathbf{u}'}{dt} = \mathbf{F}', \tag{7.3}$$

or

$$m\frac{d^2\mathbf{x}}{dt^2} = m\frac{d^2\mathbf{x}'}{dt^2}. \tag{7.4}$$

The result expressed through equation (7.4) is a statement of the **covariance** of Newton's Second Law under a Galilean transformation. More precisely, the Second Law retains the same mathematical form no matter which inertial frame is used to express it, as long as velocities transform according to the simple addition rule stated in equation (7.1).

It is also important to realize that implicit to this derivation was the fact that everywhere it was assumed that, although velocities can change from one inertial frame to another, time proceeds independently of which reference frame is used. That is, if t and t' are the time in K and K', respectively, and if they are synchronized initially such that $t = t' = 0$ then

$$t = t', \qquad \text{at all times.} \tag{7.5}$$

Although the concepts of Galilean transformation (i.e., equation (7.1)) and absolute time (i.e., equation (7.5)), and therefore Newton's Second Law, are valid for a vast domain of applications, they were eventually found to be inadequate for system where velocities approach the speed of light or for phenomenon that are electrodynamic in nature (i.e., those studied using Maxwell's equations). To correctly account for a larger proportion of physical systems, we must replace the Galilean by the Lorentz transformation, abandon the notion of absolute time, and replace the formalism of Newtonian mechanics by that of special relativity.

7.2 The Lorentz Transformation

The special theory of relativity is based on two fundamental postulates:

- **I.** The laws of nature are the same in two frames of reference in uniform relative motion with no rotation.
- **II.** The speed of light is finite and independent of the motion of its source in any frame of reference.

We consider two inertial frames K and K', as shown in Figure 7-1, with relative uniform velocity v along their respective x-axis, the origins of which are assumed to coincide at times $t = t' = 0$. If a light source at the origin and at rest in K emits a short pulse at time $t = 0$, then an observer at rest in any of the two frames of reference will (according to postulate II) see a shell of radiation centered on the origin expanding at the speed of light c. That is, a shell of radiation is observed independent of the frame of the inertial observer. Because this result obviously contradicts our previous (Galilean) assumption that velocities are additive, it forces us to reconsider our notions of space and time, and view them as quantities peculiar to each frame of reference and not universal. Therefore, we have for the equations of the expanding sphere in the frames

$$\begin{aligned} c^2t^2 - \left(x^2 + y^2 + z^2\right) &= 0 \\ c^2t'^2 - \left(x'^2 + y'^2 + z'^2\right) &= 0, \end{aligned} \tag{7.6}$$

or alternatively,

$$c^2t^2 - \left(x^2 + y^2 + z^2\right) = c^2t'^2 - \left(x'^2 + y'^2 + z'^2\right). \tag{7.7}$$

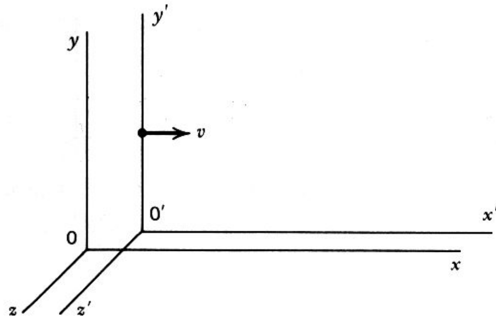

Figure 7-1 - Two inertial frames with a relative velocity v along the x-axis.

After consideration of the homogeneity and isotropic nature of space-time, it can be shown that equation (7.7) applies in general, i.e., not only to light propagation. The coordinate transformation that satisfies this condition, and the postulates of special relativity, is the so-called **Lorentz Transformation**.

We can provide a mathematical derivation of the Lorentz transformation for the system shown in Figure 7-1 as follows (please note that a much more thorough and satisfying derivation will be found, by the more adventurous reader, in the fourth problem list). Because of the homogeneity of space-time, we will assume that the different components x_μ and x'_ν of the two frames are linked by a set of linear relations. For example, we write

$$\begin{aligned} x'_0 &= Ax_0 + Bx_1 + Cx_2 + Dx_3 \\ x'_1 &= Ex_0 + Fx_1 + Gx_2 + Hx_3, \end{aligned} \tag{7.8}$$

and similar equations for x'_2 and x'_3, where we introduced the following commonly used notation $x_0 = ct$, $x_1 = x$, $x_2 = y$, and $x_3 = z$. However, since the two inertial frames exhibit a relative motion only along the x-axis, we will further assume that the directions perpendicular to the direction of motion are the same for both systems with

$$\begin{aligned} x'_2 &= x_2 \\ x'_3 &= x_3. \end{aligned} \tag{7.9}$$

Furthermore, because we consider that these perpendicular directions should be unchanged by the relative motion, and that at low velocity (i.e., when $v \ll c$) we must have

$$x'_1 = x_1 - vt, \tag{7.10}$$

we will also assume that the transformations do not "mix" the parallel and perpendicular components. That is, we set $C = D = G = H = 0$ and simplify equations (7.8) to

$$\begin{aligned} x_0' &= Ax_0 + Bx_1 \\ x_1' &= Ex_0 + Fx_1. \end{aligned} \tag{7.11}$$

Therefore, we only need to solve for the relationship between (x_0', x_1') and (x_0, x_1). To do so, we first consider a particle that is at rest at the origin of the referential K such that $x_1 = 0$ and its velocity as seen by an observer at rest in K' is $-v$. Using equations (7.11) we find that

$$\frac{x_1'}{x_0'} = -\frac{v}{c} = \frac{E}{A}. \tag{7.12}$$

Second, we consider a particle at rest at the origin of K' such that now $x_1' = 0$ and its velocity as seen in K is v. This time we find from equations (7.11) that

$$\frac{x_1}{x_0} = \frac{v}{c} = -\frac{E}{F}, \tag{7.13}$$

and the combination of equations (7.12) and (7.13) shows that $A = F$; we rewrite equations (7.11) as

$$\begin{aligned} x_0' &= A\left(x_0 + \frac{B}{A}x_1\right) \\ x_1' &= A\left(x_1 - \frac{v}{c}x_0\right). \end{aligned} \tag{7.14}$$

Third, we note that from Postulate II the propagation of a light pulse must happen at the speed of light in *both* inertial frames. We then set $x_0' = x_1'$ and $x_0 = x_1$ in equations (7.14) to find that

$$\frac{B}{A} = -\frac{v}{c}, \tag{7.15}$$

and

$$\begin{aligned} x_0' &= A\left(x_0 - \frac{v}{c}x_1\right) \\ x_1' &= A\left(x_1 - \frac{v}{c}x_0\right). \end{aligned} \tag{7.16}$$

Evidently, we could have instead proceeded by first expressing the x_μ as a function of the x_ν' with

$$
\begin{aligned}
x_0 &= A'x_0' + B'x_1' \\
x_1 &= E'x_0' + F'x_1',
\end{aligned}
\tag{7.17}
$$

from which, going through the same process as above, we would have found that

$$
\begin{aligned}
x_0 &= A'\left(x_0' + \frac{v}{c} x_1' \right) \\
x_1 &= A'\left(x_1' + \frac{v}{c} x_0' \right).
\end{aligned}
\tag{7.18}
$$

Not surprisingly, equations (7.18) are similar in form to equations (7.16) with v replaced by $-v$. The first postulate of special relativity tells us, however, that the laws of physics must be independent of the inertial frame. This implies that

$$
A = A' \tag{7.19}
$$

(this result can also be verified by inserting equations (7.16) and (7.18) into equation (7.7), as this will yield $A^2 = A'^2$). If we insert equations (7.18) into equations (7.16) we find that

$$
A = \left[1 - \left(\frac{v}{c} \right)^2 \right]^{-1/2}. \tag{7.20}
$$

we can finally write the Lorentz transformation, in its usual form for the problem at hand, as

$$
\begin{aligned}
x_0' &= \gamma\left(x_0 - \beta x_1 \right) \\
x_1' &= \gamma\left(x_1 - \beta x_0 \right) \\
x_2' &= x_2 \\
x_3' &= x_3,
\end{aligned}
\tag{7.21}
$$

with

$$
\begin{aligned}
\boldsymbol{\beta} &= \frac{\mathbf{v}}{c} \\
\beta &= |\boldsymbol{\beta}| \\
\gamma &= \left(1 - \beta^2 \right)^{-1/2}.
\end{aligned}
\tag{7.22}
$$

The inverse transformation is easily found by swapping the two sets of coordinates, and by changing the sign of the velocity. We then get

$$
\begin{aligned}
x_0 &= \gamma\left(x_0' + \beta x_1'\right) \\
x_1 &= \gamma\left(x_1' + \beta x_0'\right) \\
x_2 &= x_2' \\
x_3 &= x_3'.
\end{aligned}
\tag{7.23}
$$

Alternatively, it should be noted that equations (7.21) could be expressed with a single matrix equation relating the coordinates of the two inertial frames

$$
\vec{\mathbf{x}}' = \mathbf{L}^x\left(\beta\right)\vec{\mathbf{x}}, \tag{7.24}
$$

where the arrow is used for space-time vectors and distinguishes them from ordinary space vectors. More explicitly, this matrix equation is written as

$$
\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{7.25}
$$

It is easy to verify that the inverse of the matrix present in equation (7.25) is that which can similarly be obtained from equations (7.23). Although these transformations apply to frames that have their respective system of axes aligned with each other, Lorentz transformations between two arbitrarily oriented systems with a general relative velocity $\mathbf{v}$ can be deduced by starting with equations (7.21) (or (7.23)) and apply the needed spatial rotations.

For example, to find the Lorentz transformation $\mathbf{L}(\boldsymbol{\beta})$ applicable when the axes for K and K' remain aligned to each other, but the relative velocity $\mathbf{v}$ is allowed to take on an arbitrary orientation, we could first make a rotation $\mathbf{R}$ that will bring the x_1-axis parallel to the orientation of the velocity vector, then follow this with the basic Lorentz transformation $\mathbf{L}^x\left(\beta\right)$ defined by equations (7.25), and finish by applying the inverse rotation $\mathbf{R}^{-1}$. That is,

$$
\mathbf{L}(\boldsymbol{\beta}) = \mathbf{R}^{-1}\mathbf{L}^x\left(\beta\right)\mathbf{R}. \tag{7.26}
$$

When these operations are performed, one then finds (see the fourth problem list)

$$
\begin{aligned}
x_0' &= \gamma\left(x_0 - \boldsymbol{\beta}\cdot\mathbf{x}\right) \\
\mathbf{x}' &= \mathbf{x} + \frac{\left(\gamma - 1\right)}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} - \gamma\,\boldsymbol{\beta}x_0.
\end{aligned}
\tag{7.27}
$$

Since the Lorentz transformation "mixes" space and time coordinates between the two frames, we cannot arbitrarily dissociate the two types of coordinates. The basic unit in space-time is now an **event**, which is specified by a location in space and time given in relation to any system of reference. This mixture of space and time makes it evident that we must abandon our cherished and intuitive notion of absolute time.

7.2.1 **Four-vectors**

Since we saw in the last section that the Lorentz transformation can simply be expressed as a matrix relating the coordinates of two frames of reference (see equation (7.25)), it was natural to define vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{x}}'$ to represent these coordinates. Because these vectors have for components, they are called **four-vectors**. Just like the coordinate four-vector contains a four coordinates $\left(x_0, x_1, x_2, x_3\right)$, an arbitrary four-vector $\vec{\mathbf{A}}$ has four components $\left(A_0, A_1, A_2, A_3\right)$. Moreover, the "invariance" of the four-vector $\vec{\mathbf{x}}$ expressed through equation (7.7) is also applicable to any other four-vectors. That is, a four-vector must obey the following relation

$$A_0^2 - (A_1^2 + A_2^2 + A_3^2) = A_0'^2 - (A_1'^2 + A_2'^2 + A_3'^2). \tag{7.28}$$

7.2.2 **Proper Time and Time Dilation**

We define the infinitesimal invariant ds associated with the infinitesimal coordinate four-vector $d\vec{\mathbf{x}}$ as (see equation (7.7))

$$\begin{aligned} ds^2 &= dx_0^2 - \left(dx_1^2 + dx_2^2 + dx_3^2\right) \\ &= \left(cdt\right)^2 - \left(dx^2 + dy^2 + dz^2\right). \end{aligned} \tag{7.29}$$

If a inertial frame K' is moving relative to another one (K) with a velocity $\mathbf{v}$ such that $d\mathbf{x} = \mathbf{v}\,dt$ (with $d\mathbf{x}$ the spatial part of $d\vec{\mathbf{x}}$), then equation (7.29) can be written as

$$\begin{aligned} ds^2 &= c^2dt^2 - \left|d\mathbf{x}\right|^2 \\ &= c^2dt^2\left(1-\beta^2\right). \end{aligned} \tag{7.30}$$

For an observer at rest in K', however, it must be (by definition) that $d\mathbf{x}' = 0$, and from equation (7.30)

$$ds^2 = c^2dt'^2 = c^2dt^2\left(1-\beta^2\right). \tag{7.31}$$

We define the **proper time** τ with

$$\boxed{ds = cd\tau} \tag{7.32}$$

Comparison of equations (7.31) with (7.32) shows that an element of proper time is the actual time interval measured with a clock at rest in a system. The elements of time elapsed in each reference frame (and measured by observers at rest in each of them) are related through

$$d\tau = dt\sqrt{1-\beta^2} = \frac{dt}{\gamma}. \tag{7.33}$$

Since $\gamma > 1$, then to an observer at rest in K time appears to be passing by more slowly in K'. More precisely, a proper time interval $\tau_2 - \tau_1$ will be seen in K as lasting

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1-\beta^2(\tau)}} = \int_{\tau_1}^{\tau_2} \gamma(\tau)\, d\tau, \tag{7.34}$$

where we assumed that the velocity could be changing with (proper) time. This phenomenon is called **time dilation**.

7.2.3 Length Contraction

Let us suppose that a rod of length L_0 is kept at rest in K', and laid down in the x' direction. We now inquire as to what will be the length for this rod when measured by an observer in K. One important thing to realize is that the length of the rod as measured in K will be $L = x_1(b) - x_1(a)$, where $x_1(a)$ and $x_1(b)$ are the position of the ends of the rod at the *same time* t when the measurement occurs. More precisely, t is the time coordinate associated with K and no other frame. So, from equations (7.21) we have

$$\begin{aligned} L_0 &\equiv x_1'(b) - x_1'(a) \\ &= \gamma\left[x_1(b) - \beta x_0\right] - \gamma\left[x_1(a) - \beta x_0\right] \\ &= \gamma\left[x_1(b) - x_1(a)\right], \end{aligned} \tag{7.35}$$

or alternatively

$$L = \frac{L_0}{\gamma}. \tag{7.36}$$

Therefore, to an observer in K the rod appears to be smaller than the length it has when measured at rest (i.e., in K'). This apparently peculiar result is just a consequence of the fact that, in special relativity, events that are simultaneous (i.e., happen at the same time) in one reference frame will not be in another (if the two frames are moving relative to one another). *The concept of simultaneity must be abandoned in special relativity.*

7.2.4 **Relativistic Doppler Shift**

We saw in section 7.2 that the space-time "length" element ds is invariant as one goes from one inertial frame to the next. Similarly, since the number of crests in a wave train can be, in principle, counted, it must be a relativistic invariant. Then, the same must be true of the phase of a plane wave ϕ. Mathematically, if ω and $\mathbf{k}$ are, respectively, the angular frequency and the wave vector of a plane wave as measured in K, then the invariance of the phase means that

$$\phi = \omega t - \mathbf{k} \cdot \mathbf{x} = \omega' t' - \mathbf{k}' \cdot \mathbf{x}'. \tag{7.37}$$

Since $\omega = ck$ and with $\mathbf{k} = k\mathbf{n}$, we can write

$$k\left(ct - \mathbf{n} \cdot \mathbf{x}\right) = k'\left(ct' - \mathbf{n}' \cdot \mathbf{x}'\right), \tag{7.38}$$

and upon using equations (7.27), while breaking all vectors in parts parallel and perpendicular to the velocity (i.e., $\mathbf{x} = \mathbf{x}_{\|} + \mathbf{x}_{\perp}$), we have

$$\begin{aligned} k\left(ct - \mathbf{n}_{\|} \cdot \mathbf{x}_{\|} - \mathbf{n}_{\perp} \cdot \mathbf{x}_{\perp}\right) &= k'\left[\gamma\left(ct - \boldsymbol{\beta} \cdot \mathbf{x}_{\|}\right) - \gamma \mathbf{n}'_{\|} \cdot \left(\mathbf{x}_{\|} - \boldsymbol{\beta} ct\right) - \mathbf{n}'_{\perp} \cdot \mathbf{x}_{\perp}\right] \\ &= k'\left[\gamma ct\left(1 + \mathbf{n}'_{\|} \cdot \boldsymbol{\beta}\right) - \gamma \mathbf{x}_{\|} \cdot \left(\mathbf{n}'_{\|} + \boldsymbol{\beta}\right) - \mathbf{n}'_{\perp} \cdot \mathbf{x}_{\perp}\right]. \end{aligned} \tag{7.39}$$

If this equation is to hold at all times t and position $\mathbf{x}$, then the different coefficients for t, $\mathbf{x}_{\|}$, and $\mathbf{x}_{\perp}$ on either side must be equal. That is,

$$\begin{aligned} \omega &= \gamma \omega'\left(1 + \mathbf{n}'_{\|} \cdot \boldsymbol{\beta}\right) \\ k_{\|} &= \gamma\left(k'_{\|} + \beta \frac{\omega'}{c}\right) \\ \mathbf{k}_{\perp} &= \mathbf{k}'_{\perp}. \end{aligned} \tag{7.40}$$

The first of equations (7.40) is that for the Doppler shift. It is important to note that the term in parentheses is the same as the one that appears in the non-relativistic version of the formula, but that there will also be a Doppler shift in the relativistic case (although of second order) even if the direction of propagation of the wave is perpendicular to velocity vector. This is because of the presence of the γ factor on the right-hand side of this equation. Equally important is the fact that, as evident from our analysis, the frequency and the wave vector form a four-vector $\vec{\mathbf{k}}$ (with $k_0 = \omega/c$), and that the phase ϕ is an invariant resulting from the following scalar product

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{x}} = \phi, \tag{7.41}$$

from equation (7.37). The second of equations (7.40) can be transformed to give

$$
\begin{aligned}
\cos(\theta) &= \frac{k'}{k}\gamma\left[\cos(\theta')+\beta\right] \\
&= \frac{\omega'}{\omega}\gamma\left[\cos(\theta')+\beta\right] \\
&= \frac{\cos(\theta')+\beta}{1+\beta\cos(\theta')},
\end{aligned}
\tag{7.42}
$$

where the first of equations (7.40) was used, and θ and θ' are the angles of $\mathbf{k}$ and $\mathbf{k}'$ relative to $\boldsymbol{\beta}$. In turn, the last two of equations (7.40) can be combined as follows

$$
\begin{aligned}
\frac{|\mathbf{k}_\perp|}{k_\parallel} &= \frac{k'_\perp}{\gamma\left(k'_\parallel+\beta k'_0\right)} \\
\tan(\theta) &= \frac{\sin(\theta')}{\gamma\left[\cos(\theta')+\beta\right]}.
\end{aligned}
\tag{7.43}
$$

Finally, inverting equation (7.42) to get $\cos(\theta')$ as a function of $\cos(\theta)$ and $\sin(\theta)$, substituting it into the first of equations (7.40) will easily lead to the following set of equations

$$
\begin{aligned}
\omega' &= \gamma\omega\left(1-\mathbf{n}_\parallel\cdot\boldsymbol{\beta}\right) \\
k'_\parallel &= \gamma\left(k_\parallel-\beta k_0\right) \\
\mathbf{k}'_\perp &= \mathbf{k}_\perp.
\end{aligned}
\tag{7.44}
$$

This result is equivalent to that of, and could have been obtained in the same manner as was done for, equations (7.40). It should also be noted that for a plane wave

$$
\vec{\mathbf{k}}\cdot\vec{\mathbf{k}} = 0. \tag{7.45}
$$

7.2.5 The Transformation of Velocities

We now endeavor to find out what will the velocity $\mathbf{u}$ of a particle as measured in K be if it has a velocity $\mathbf{u}'$ in K'. From equations (7.27), we can write

$$
\begin{aligned}
u_\parallel &= \frac{dx_\parallel}{dt} = \frac{\gamma\left(dx'_\parallel+v\,dt'\right)}{\gamma\left(dt'+\frac{\mathbf{v}\cdot d\mathbf{x}'}{c^2}\right)} = \frac{u'_\parallel+v}{1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}} \\
\mathbf{u}_\perp &= \frac{d\mathbf{x}_\perp}{dt} = \frac{d\mathbf{x}'_\perp}{\gamma\left(dt'+\frac{\mathbf{v}\cdot d\mathbf{x}'}{c^2}\right)} = \frac{\mathbf{u}'_\perp}{\gamma\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)},
\end{aligned}
\tag{7.46}
$$

and it is easily seen that when both u and v are much smaller than c, equations (7.46) reduces to the usual Galilean addition of velocities $\mathbf{u}' + \mathbf{v}$. The orientation of $\mathbf{u}$, specified by its spherical coordinate angles θ and φ, can be determined as follows

$$\begin{aligned} \tan(\theta) &= \frac{u_\perp}{u_\parallel} = \frac{u' \sin(\theta')}{\gamma\left[u' \cos(\theta') + v\right]} \\ \varphi &= \varphi', \end{aligned} \tag{7.47}$$

since $u_2/u_3 = u_2'/u_3'$.

7.2.6 The Four-velocity and the Four-momentum

We already know that $d\vec{\mathbf{x}}$ is a four-vector. Since we should expect that the division of a four-vector by an invariant would not change its character (i.e., the result will still be a four-vector), then an obvious candidate for a four-vector is the **four-velocity**

$$\boxed{\vec{\mathbf{U}} \equiv \frac{d\vec{\mathbf{x}}}{d\tau}} \tag{7.48}$$

The components of the four-velocity are easily evaluated with

$$\boxed{\begin{aligned} U_0 &= \frac{dx_0}{dt}\frac{dt}{d\tau} = \gamma_u c \\ \mathbf{U} &= \frac{d\mathbf{x}}{dt}\frac{dt}{d\tau} = \gamma_u \mathbf{u} \end{aligned}} \tag{7.49}$$

with $\gamma_u = \left(1 - u^2/c^2\right)^{-\frac{1}{2}}$. Thus the time component of the four-velocity is γ_u times c, while the spatial part is γ_u times the ordinary velocity. The transformation of the four-velocity under a Lorentz transformation is (using as always the inertial frames K and K' defined earlier; that is to say, we are now considering the case of a particle traveling at a velocity $\mathbf{u}$ in K, which in turn has a velocity $-\mathbf{v}$ relative to K')

$$\begin{aligned} U_0' &= \gamma_v\left(U_0 - \boldsymbol{\beta} \cdot \mathbf{U}\right) \\ U_\parallel' &= \gamma_v\left(U_\parallel - \beta U_0\right) \\ \mathbf{U}_\perp' &= \mathbf{U}_\perp, \end{aligned} \tag{7.50}$$

with $\gamma_v = \left(1 - \beta^2\right)^{-\frac{1}{2}}$. Using equations (7.49) for the definition of the four-velocity, and inserting it into equations (7.50), we get

$$\begin{aligned}\gamma_{u'}c &= \gamma_v\gamma_u\left(c - \boldsymbol{\beta}\cdot\mathbf{u}\right)\\ \gamma_{u'}u'_{\parallel} &= \gamma_v\gamma_u\left(u_{\parallel} - \beta c\right)\\ \gamma_{u'}\mathbf{u}'_{\perp} &= \gamma_u\mathbf{u}_{\perp}.\end{aligned} \tag{7.51}$$

The first of this set of equations can be rewritten to express the transformation of velocities in terms of the γ's

$$\gamma_{u'} = \gamma_v\gamma_u\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right), \tag{7.52}$$

while dividing the second and third by the first of equations (7.51) yields

$$\begin{aligned}u'_{\parallel} &= \frac{u_{\parallel} - v}{1 - \dfrac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\\ \mathbf{u}'_{\perp} &= \frac{\mathbf{u}_{\perp}}{\gamma_v\left(1 - \dfrac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)}.\end{aligned} \tag{7.53}$$

It will be recognized that this last result expresses, just like equations (7.46), the law for the addition of relativistic velocities. Another important quantity associated with $\vec{\mathbf{U}}$ is the following relativistic invariant

$$\vec{\mathbf{U}}\cdot\vec{\mathbf{U}} = \gamma_u^2\left(c^2 - \mathbf{u}\cdot\mathbf{u}\right) = c^2. \tag{7.54}$$

The **four-momentum** $\vec{\mathbf{P}}$ of a particle is another fundamental relativistic quantity. It is simply related to the four-velocity by

$$\boxed{\vec{\mathbf{P}} = m\vec{\mathbf{U}}} \tag{7.55}$$

where m is the mass of the particle. Referring to equations (7.49), we find that the components of the four-momentum are

$$\boxed{\begin{aligned}P_0 &= \gamma mc\\ \mathbf{p} &= \gamma m\mathbf{u}\end{aligned}} \tag{7.56}$$

with $\gamma = \left(1 - u^2/c^2\right)^{-1/2}$, and $\mathbf{u}$ the ordinary velocity of the particle. If we consider the expansion of P_0c for a non-relativistic velocity $u \ll c$, then we find

$$P_0 c = \gamma m c^2 = mc^2 + \frac{1}{2} m u^2 + \cdots \tag{7.57}$$

Since the second term on the right-hand side of equation (7.57) is the non-relativistic expression of the kinetic energy of the particle, we interpret $E = P_0 c$ as the total energy of the particle. The first term (i.e., mc^2) is independent of the velocity, and is interpreted as the **rest energy** of the particle.

Finally, since $\vec{\mathbf{P}}$ is a four-vector, we inquire about the Lorentz invariant that can be calculated with

$$\vec{\mathbf{P}} \cdot \vec{\mathbf{P}} = m^2 \vec{\mathbf{U}} \cdot \vec{\mathbf{U}} = m^2 c^2, \tag{7.58}$$

from equation (7.54). But since we also have

$$\vec{\mathbf{P}} \cdot \vec{\mathbf{P}} = \frac{E^2}{c^2} - \mathbf{p} \cdot \mathbf{p}, \tag{7.59}$$

we find Einstein's famous equation linking the energy of a particle to its mass

$$\begin{aligned} E^2 &= m^2 c^4 + c^2 p^2 \\ &= m^2 c^4 \left(1 + \frac{p^2}{m^2 c^2} \right) \\ &= m^2 c^4 \left(1 + \gamma^2 \frac{u^2}{c^2} \right) \\ &= \left(\gamma m c^2 \right)^2, \end{aligned} \tag{7.60}$$

or alternatively

$$\boxed{E = \gamma m c^2 = \frac{mc^2}{\sqrt{1 - \dfrac{u^2}{c^2}}}} \tag{7.61}$$

7.3 Tensor Analysis

So far, we have discussed physical quantities in the context of special relativity using four-vectors. But just like in ordinary three-space where vectors can be handled using a component notation, we would like to do the same in space-time. For example, the position vector $\mathbf{x}$ has three components x^i, with $i = 1, 2, 3$ for x, y, and z, respectively, in ordinary space, and similarly the "position" four-vector $\vec{\mathbf{x}}$ has four components x^α, with $\alpha = 0, 1, 2, 3$ for ct, x, y, and z. Please note the position of the indices (i.e., they are superscript); we call this representation for the four-position or any other four-vector the **contravariant** representation.

In ordinary space, the scalar product between, say, $\mathbf{x}$ and $\mathbf{y}$ can also be written

$$\mathbf{x}\cdot\mathbf{y}=\left(x_1,x_2,x_3\right)\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}, \tag{7.62}$$

where the vector $\mathbf{x}$ was represented as a row vector, as opposed to a column vector, and for this reason we used subscript. That is, we differentiate between the row and column representations of the same quantity by changing the position of the index identifying its components. We call subscript representation the **covariant** representation. Using these definitions, and Einstein's implied summation on repeated indices, we can write the scalar product of equation (7.62) as

$$\mathbf{x}\cdot\mathbf{y}=x_i y^i. \tag{7.63}$$

It should be noted that if the scalar product is to be invariant in three-space (e.g., $x^2=\mathbf{x}\cdot\mathbf{x}$ does not change after the axes of the coordinate system, or $\mathbf{x}$ itself, are rotated), then covariant and contravariant components must transform differently under a coordinate transformation. For example, if we apply a rotation $\mathbf{R}$ to the system of axes, then for $\mathbf{x}\cdot\mathbf{y}$ to be invariant under $\mathbf{R}$ we must have that

$$\mathbf{x}\cdot\mathbf{y}=\left[\mathbf{x}\mathbf{R}^{-1}\right]\cdot\left[\mathbf{R}\mathbf{y}\right]=\mathbf{x}'\cdot\mathbf{y}', \tag{7.64}$$

where $\mathbf{x}'$ and $\mathbf{y}'$ are the transformed vectors. It is clear from equation (7.64) that column (i.e., contravariant) vectors transform with $\mathbf{R}$, while row (i.e., covariant) vectors transform with its inverse. Similarly, in space-time we define the **covariant** representation of the four-vector $\vec{\mathbf{x}}$ (often written as $\tilde{\mathbf{x}}$) has having the components x_α, with $\alpha=0,1,2,3$. Therefore, the scalar product of two four-vectors is

$$\vec{\mathbf{x}}\cdot\vec{\mathbf{y}}=x_\alpha y^\alpha. \tag{7.65}$$

Four-vectors have only one index, and are accordingly called contravariant or covariant **tensors of rank one**. If we have two different sets of coordinates x^α and x'^α related to each other through a Lorentz transformation (i.e., a coordinate transformation)

$$x'^\alpha=x'^\alpha\left(x^0,x^1,x^2,x^3\right), \tag{7.66}$$

then any contravariant vector A^α will transform according to

$$\boxed{A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta}} \tag{7.67}$$

Conversely, a covariant vector A_{α} will transform according to

$$\boxed{A'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} A_{\beta}} \tag{7.68}$$

Of course, tensors are not limited to one index and it is possible to define tensors of different ranks. The simplest case is that of a tensor of rank zero, which corresponds to an invariant. An invariant has the same value independent of the inertial system (or coordinate system) it is evaluated in. For example, the infinitesimal space-time length ds is defined with

$$ds^2 = dx_{\alpha} dx^{\alpha}. \tag{7.69}$$

The invariance of the scalar product can now be verified with

$$\begin{aligned}\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A'_{\alpha} B'^{\alpha} &= \frac{\partial x^{\beta}}{\partial x'^{\alpha}} A_{\beta} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} B^{\gamma} \\ &= \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} A_{\beta} B^{\gamma} = \frac{\partial x^{\beta}}{\partial x^{\gamma}} A_{\beta} B^{\gamma} \\ &= \delta^{\beta}{}_{\gamma} A_{\beta} B^{\gamma} = A_{\gamma} B^{\gamma}.\end{aligned} \tag{7.70}$$

Contravariant, covariant, and mixed tensors of rank two are, respectively, defined by

$$\boxed{\begin{aligned}F'^{\alpha\beta} &= \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \\ G'_{\alpha\beta} &= \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} G_{\gamma\delta} \\ H'^{\alpha}{}_{\beta} &= \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} H^{\gamma}{}_{\delta}\end{aligned}} \tag{7.71}$$

These definitions can easily be extended to tensors of higher rank. It should be kept in mind that different flavors of a given tensor are just different representation of the same mathematical object. It should, therefore, be possible to move from one representation to another. This is achieved using the symmetric **metric tensor** $g_{\alpha\beta}$ ($g_{\alpha\beta} = g_{\beta\alpha}$). That is, the metric tensor allows for indices to be lowered as follows

$$\begin{aligned} F_{\alpha}{}^{\beta} &= g_{\alpha\gamma}F^{\gamma\beta} \\ F^{\alpha}{}_{\beta} &= g_{\beta\gamma}F^{\alpha\gamma}. \end{aligned} \tag{7.72}$$

If we define a contravariant version of the metric tensor such that

$$\boxed{g_{\alpha\gamma}g^{\gamma\beta} = g_{\alpha}{}^{\beta} = \delta_{\alpha}{}^{\beta}} \tag{7.73}$$

then indices of tensors can also be raised

$$\begin{aligned} F_{\alpha}{}^{\beta} &= g^{\beta\gamma}F_{\alpha\gamma} \\ F^{\alpha}{}_{\beta} &= g^{\alpha\gamma}F_{\gamma\beta}. \end{aligned} \tag{7.74}$$

The metric tensor can also be made to appear prominently in the definition of the scalar product. For example,

$$ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta}. \tag{7.75}$$

From this, and the definition of the space-time length element (i.e., equation (7.29)), it is deduced that the metric tensor is diagonal with

$$g_{00} = 1, \quad \text{and} \quad g_{11} = g_{22} = g_{33} = -1. \tag{7.76}$$

Furthermore, it is easy to see that $g^{00} = g_{00}$, $g^{11} = g_{11}$, etc. Please take note that equation (7.76) is only true for flat space-time in Cartesian coordinates. It would not apply, for example, if space-time were curved (in the context of general relativity), or if one used curvilinear coordinates (e.g., spherical coordinates). In flat space-time the metric tensor is often written as $\eta_{\alpha\beta}$, and called the **Minkowski metric**. From equation (7.76), we can also better see the important difference between the contravariant and covariant flavors of the same four-vector, since

$$A^{\alpha} = \begin{pmatrix} A^0 \\ \mathbf{A} \end{pmatrix}, \quad \text{and} \quad A_{\alpha} = \left(A^0, -\mathbf{A}\right). \tag{7.77}$$

Finally, we inquire about the nature of the gradient operator in tensor analysis. Since we know from elementary calculus that

$$\frac{\partial}{\partial x'^{\alpha}} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}}\frac{\partial}{\partial x^{\beta}}, \tag{7.78}$$

then *the components of the gradient operator transform as that of a covariant vector*. Since, from equations (7.74), we can write

$$x_\alpha = g_{\alpha\beta} x^\beta, \tag{7.79}$$

then

$$\frac{\partial}{\partial x^\gamma} = \frac{\partial x_\alpha}{\partial x^\gamma} \frac{\partial}{\partial x_\alpha} = \frac{\partial \left(g_{\alpha\beta} x^\beta \right)}{\partial x^\gamma} \frac{\partial}{\partial x_\alpha}, \tag{7.80}$$

and, since in flat space-time the metric tensor is constant,

$$\begin{aligned} \frac{\partial}{\partial x^\gamma} &= \left[x^\beta \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + g_{\alpha\beta} \frac{\partial x^\beta}{\partial x^\gamma} \right] \frac{\partial}{\partial x_\alpha} \\ &= g_{\alpha\beta} \delta_\gamma^\beta \frac{\partial}{\partial x_\alpha} \\ &= g_{\alpha\gamma} \frac{\partial}{\partial x_\alpha}. \end{aligned} \tag{7.81}$$

Alternatively, operating on both sides of equation (7.81) with $g'^{\beta\gamma}$ (while using the "prime" coordinates) yields a gradient operator that transforms as a contravariant tensor since (using the first of equations (7.71) and equation (7.78))

$$\begin{aligned} \frac{\partial}{\partial x'_\alpha} = g'^{\alpha\gamma} \frac{\partial}{\partial x'^\gamma} &= \left(g^{\mu\nu} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\gamma}{\partial x^\nu} \right) \left(\frac{\partial x^\beta}{\partial x'^\gamma} \frac{\partial}{\partial x^\beta} \right) \\ &= g^{\mu\nu} \frac{\partial x'^\alpha}{\partial x^\mu} \left(\frac{\partial x'^\gamma}{\partial x^\nu} \frac{\partial x^\beta}{\partial x'^\gamma} \right) \frac{\partial}{\partial x^\beta} = g^{\mu\nu} \frac{\partial x'^\alpha}{\partial x^\mu} \delta_\nu{}^\beta \frac{\partial}{\partial x^\beta} \\ &= g^{\mu\beta} \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial}{\partial x^\beta} = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial}{\partial x_\mu}. \end{aligned} \tag{7.82}$$

In what will follow we will use the notation defined below for partial derivatives

$$\boxed{\begin{aligned} \partial^\alpha &\equiv \frac{\partial}{\partial x_\alpha} = \begin{pmatrix} \partial/\partial x^0 \\ -\nabla \end{pmatrix} \\ \partial_\alpha &\equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \nabla \right) \end{aligned}} \tag{7.83}$$

The four-dimensional Laplacian operator $\Box$ is the invariant defined by the scalar vector

$$\boxed{\Box \equiv \partial_\alpha \partial^\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2} \tag{7.84}$$

We recognize the wave equation operator. It is important to realize that equations (7.80) and (7.82) are tensors only in flat space-time (i.e., no curvature, and using Cartesian coordinates). The corresponding equations for the general case will differ.

7.4 Covariance of Electrodynamics

Before we discuss the invariance of the equations of electrodynamics under Lorentz transformations, let us rewrite these equations using Gaussian units. For the Maxwell equations we have

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{D} &= 4\pi\rho \\ \nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} &= \frac{4\pi}{c}\mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \end{aligned}} \tag{7.85}$$

We should note that with these units $\varepsilon_0 = \mu_0 = 1$, and in free space $\varepsilon = \mu = 1$ such that $\mathbf{D} = \varepsilon\mathbf{E} = \mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H} = \mathbf{H}$. Also important are the equation for the Lorentz force

$$\boxed{\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)} \tag{7.86}$$

and the continuity equation

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0} \tag{7.87}$$

which is left unchanged from its form in SI units.

Besides the electromagnetic fields, two other quantities appear in Maxwell's equations: the speed of light c and the charge q. We already know that the speed of light is an invariant (from the second postulate of special relativity). Experiments show that the charge is also an invariant (for example, a speed dependency of the charge would imply changes in the net charge in a piece of material with chemical reaction; there is no experimental evidence for this), and this has implications for what follows. Consider, for example, the four-volume element d^4x defined as

$$d^4x = dx^0 dx^1 dx^2 dx^3, \tag{7.88}$$

which transforms as follows under a Lorentz transformation

$$d^4x' = dx'^0 dx'^1 dx'^2 dx'^3 = \left|L^x(\beta)\right| dx^0 dx^1 dx^2 dx^3, \tag{7.89}$$

where $\left|L^x(\beta)\right|$ is the (Jacobian) determinant of the Lorentz transformation matrix (see equation (7.25)). It is straightforward to show that $\left|L^x(\beta)\right| = 1$, and therefore that the four-volume element is an invariant since $d^4x = d^4x'$. If the charge element dq is to be an invariant with

$$dq = \rho dx^1 dx^2 dx^3, \tag{7.90}$$

then the charge density must transform in the same manner as the time component of a four-vector. Thus, if we assume that $J^0 = \rho c$ is the time component of a four-vector $\vec{\mathbf{J}}$ (the **four-current**), and we operate on it with the gradient operator of equation (7.83), then in consideration of equation (7.87) it must be that the space part is the current density $\mathbf{J}$, and

$$\boxed{J^\mu = \begin{pmatrix} \rho c \\ \mathbf{J} \end{pmatrix}} \tag{7.91}$$

The continuity equation is now written as

$$\partial_\alpha J^\alpha = 0. \tag{7.92}$$

We next look at the vector and scalar potentials in the Lorentz gauge where (using Gaussian units) we have

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} &= \frac{4\pi}{c}\mathbf{J} \\ \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi &= 4\pi\rho, \end{aligned} \tag{7.93}$$

and

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0. \tag{7.94}$$

If we define the **four-potential** A^μ

$$\boxed{A^{\mu} = \begin{pmatrix} \Phi \\ \mathbf{A} \end{pmatrix}} \tag{7.95}$$

then equations (7.93) and (7.94) can be written in a covariant form as

$$\Box A^{\mu} = \frac{4\pi}{c} J^{\mu}, \tag{7.96}$$

and

$$\partial_{\mu} A^{\mu} = 0. \tag{7.97}$$

Since the electromagnetic fields can be obtained from the potentials with (again using Gaussian units)

$$\begin{aligned} \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi \\ \mathbf{B} &= \nabla \times \mathbf{A}, \end{aligned} \tag{7.98}$$

it can be shown that the components of $\mathbf{E}$ and $\mathbf{B}$ can be put together in the antisymmetric second rank **electromagnetic tensor** $F^{\alpha\beta}$

$$\boxed{F^{\alpha\beta} \equiv \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha}} \tag{7.99}$$

The components of the electromagnetic tensors are explicitly

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{pmatrix}, \tag{7.100}$$

or, alternatively, in its fully covariant form

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix}. \tag{7.101}$$

Note that E^i, for $i = x,y,z$, are the Cartesian components of $\mathbf{E}$, etc. That $F^{\alpha\beta}$ has the desired form can be verified by attempting to express the Maxwell equations with it. A little calculation will easily show that the first two of equations (7.85) can be written as

$$\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta} \tag{7.102}$$

while the last two of equations (7.85) become

$$\boxed{\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0} \tag{7.103}$$

Finally, to complete this section, we would like to express the Lorentz force in a covariant form. To do so, we must first introduce two more four-vectors. First, just as we defined in section 7.2.6 the four-velocity as the proper time derivative of $d\vec{\mathbf{x}}$, we extend this process to define the **four-acceleration** a^μ as

$$\boxed{a^\mu \equiv \frac{dU^\mu}{d\tau}} \tag{7.104}$$

It is easy to see that the space part of the four-acceleration equals the ordinary acceleration in the non-relativistic limit. Note also that the four-acceleration is *orthogonal* to the four-velocity

$$\begin{aligned} a^\mu U_\mu &= \frac{dU^\mu}{d\tau} U^\mu = \frac{1}{2} \frac{d}{d\tau}\left(U^\mu U_\mu\right) \\ &= \frac{1}{2} \frac{d\left(c^2\right)}{d\tau} = 0. \end{aligned} \tag{7.105}$$

From equation (7.104), it is a small step to define the **four-force** F^μ as

$$\boxed{F^\mu \equiv \frac{dP^\mu}{d\tau} = m\frac{dU^\mu}{d\tau} = ma^\mu} \tag{7.106}$$

The Lorentz four-force should involve the electromagnetic fields through $F^{\alpha\beta}$ and the velocity of the charge through U^γ. The simplest of such relations is

$$\boxed{\frac{dP^\alpha}{d\tau} = m\frac{dU^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_\beta} \tag{7.107}$$

It can be verified that this relation indeed satisfies equation (7.86). Furthermore, it is seen that the time component of equation (7.107) yields

$$\frac{dW}{dt} = q\mathbf{E} \cdot \mathbf{u}, \tag{7.108}$$

which is just a statement of the conservation of energy. In other words, the rate of change of particle energy W is the mechanical work done on the particle by the electric field.

7.5 Transformation of the Electromagnetic Fields

As we as previously seen, the $\mathbf{E}$ and $\mathbf{B}$ fields themselves are not tensor quantities, but components of the electromagnetic tensor $F^{\alpha\beta}$ (see equation (7.100)). Therefore, if we want to inquire about how the electromagnetic fields transform under a Lorentz transformation, we need to study the transformation of $F^{\alpha\beta}$. Going back to the first of equations (7.71), we write

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}. \tag{7.109}$$

We should note that if we define a matrix $\Lambda^{\alpha}{}_{\gamma}$ such that

$$\Lambda^{\alpha}{}_{\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}}, \tag{7.110}$$

then equation (7.109) is simply written in a matrix form as

$$\mathbf{F}' = \mathbf{\Lambda} \mathbf{F} \mathbf{\Lambda}^{\mathrm{T}}, \tag{7.111}$$

with $\mathbf{\Lambda}^{\mathrm{T}}$ the transpose of $\mathbf{\Lambda}$. In cases where the axes of the two coordinate systems are aligned but their relative velocity is arbitrarily oriented, one can start with equations (7.27) for the contravariant coordinates

$$\begin{aligned} x'^{0} &= \gamma\left(x^{0} + \beta_{i} x^{i}\right) \\ x'^{i} &= x^{i} - \frac{(\gamma-1)}{\beta^{2}} \beta^{i} \beta_{j} x^{j} - \gamma \beta^{i} x^{0} \end{aligned} \tag{7.112}$$

(please note the differences between equations (7.27) and (7.112)) in order to evaluate the partial derivatives

$$\begin{aligned} \frac{\partial x'^{0}}{\partial x^{\gamma}} &= \gamma \delta^{0}{}_{\gamma} + \gamma \beta_{i} \delta^{i}{}_{\gamma} \\ \frac{\partial x'^{i}}{\partial x^{0}} &= -\gamma \beta^{i} \\ \frac{\partial x'^{i}}{\partial x^{j}} &= \delta^{i}{}_{j} - \frac{(\gamma-1)}{\beta^{2}} \beta_{j} \beta^{i}. \end{aligned} \tag{7.113}$$

Inserting equations (7.113) (twice) in equation (7.109) will yield the components of $\mathbf{E}'$ and $\mathbf{B}'$ through

$$\begin{aligned} F'^{j0} &= E'^{j} \\ F'^{ij} &= -\varepsilon^{ijk} B'_k, \end{aligned} \tag{7.114}$$

where $B_k \equiv -B^k$, and the Levi-Civita symbol is defined such that

$$\varepsilon_{ijk} = -\varepsilon^{ijk} = \begin{cases} +1, & \text{for any even permutation of } i=1, j=2, \text{ and } k=3 \\ -1, & \text{for any odd permutation of } i=1, j=2, \text{ and } k=3 \\ 0, & \text{if any two indices are equal} \end{cases} \tag{7.115}$$

After all the required manipulations are performed, one finds that

$$\begin{aligned} \mathbf{E}' &= \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma+1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}) \\ \mathbf{B}' &= \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}). \end{aligned} \tag{7.116}$$

[To successfully derive equations (7.116) you will need to use the following

$$\begin{aligned} B_k &= \frac{1}{2} \varepsilon_{ijk} F^{ij} \\ \beta_i F^{ij} &= -[\boldsymbol{\beta} \times \mathbf{B}]^j \\ \beta_i \beta_j F^{ij} &= \boldsymbol{\beta} \cdot (\boldsymbol{\beta} \times \mathbf{B}) = 0 \\ \varepsilon_{ijk} \beta^i F^{j0} &= [\boldsymbol{\beta} \times \mathbf{E}]^k \\ \beta_m \varepsilon_{ijk} \beta^i F^{jm} &= [\boldsymbol{\beta} \times (\boldsymbol{\beta} \times \mathbf{B})]^k. \end{aligned} \tag{7.117}$$

Note that these equations (as well as equations (7.114)) involve space vectors, not tensors (i.e., four-vectors), and are therefore not valid tensor equations.]

For the simple case where the relative velocity of the aligned inertial systems is directed along the x-axis (as was the case for the K and K' systems defined earlier in the chapter), the matrix $\boldsymbol{\Lambda}$ is given by equation (7.25), and a straightforward multiplication of matrices (or the corresponding simplification of equations (7.116)) yields

$$\begin{aligned} E'_1 &= E_1 & B'_1 &= B_1 \\ E'_2 &= \gamma(E_2 - \beta B_3) & B'_2 &= \gamma(B_2 + \beta E_3) \\ E'_3 &= \gamma(E_3 + \beta B_2) & B'_3 &= \gamma(B_3 - \beta E_2) \end{aligned} \tag{7.118}$$

It should now be obvious from equations (7.116) and (7.118) that the electromagnetic fields are not independent entities but are completely interrelated through the transformation of the electromagnetic tensor. Finally, we note that we can calculate the following invariants from the electromagnetic tensor

$$F_{\alpha\beta}F^{\alpha\beta} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right), \tag{7.119}$$

and

$$\det\left(F^{\alpha\beta}\right) = \det\left(F'^{\alpha\beta}\right) = \left(\mathbf{E}\cdot\mathbf{B}\right)^2 . \tag{7.120}$$